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Reliability Analysis of a Multi-State System Using Multi-Valued Logic

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Abstract: *This paper employs multi-valued logic in the reliability analysis of a multi-state system. The paper expresses each instance of the multi-valued output of the system as an explicit function of the multi-valued inputs of the system. The various expressions are then compiled in a Multi-Valued Karnaugh Map (MVKM) which serves as the natural map for a multi-state system. The paper demonstrates its proposed technique in terms of a standard commodity-supply system, and obtains numerical results that exactly agree with those obtained by earlier methods. As a bonus, the paper utilizes the MVKM representation of the solved coherent multi-state system to illustrate its features of causality, monotonicity and relevancy.*
Keywords: *Multi-State system, Reliability analysis, Multi-valued logic, k-out-of-n system, Multi-valued Karnaugh map.*

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I. Introduction

We propose a novel method for the reliability analysis of a multi-state system *via* multi-valued logic. This method seems to be the most natural and direct way to achieve this purpose, since it seeks to express each instance of the multi-state system as an explicit function of the multi-valued inputs of the system. The ultimate output of the method is an aggregated or collective tabulation of the resulting functions. This tabulation is conveniently achieved as a Multi-Valued Karnaugh Map (MVKM), which serves as a natural, unique, and complete representation of the multi-state system.

The paper presents a detailed analysis of a standard commodity supply multi-state system, and provides a complete solution for it in MVKM form. This manually-obtained solution agrees exactly with other solutions obtained earlier *via* automated techniques [1, 2] or *via* different manual techniques [3, 4].To construct our MVKM complete solutions, we could have followed (among several possibilities) either of the following two options.

Option 1 is to use exhaustive enumeration, *i.e.*, to decide the entry (output value) for each of the individual MVKM cells. This is a very time-consuming brute-force option that does not benefit of possible similarities, shortcuts and/or aggregations. However, it decomposes the initial problem into individual problems, in which a verbal statement of the problem together with full specification of the individual inputs lead immediately to a selection of the output value from amongst its possible multitude of values.

Option 2 is to use "binary" entities to relate each of the instances of the output to multi-valued inputs. These binary inputs could be either algebraic quantities or Conventional Karnaugh Maps (CKMs) [5-13].We use the choice of algebraic quantities herein and delegate the choice of CKMs to another forthcoming paper [14].

The organization of the remainder of this paper is as follows. Section II gives a verbal and a mathematical description of the problem to be solved. Section III presents the mathematical derivation for the multi-valued output. Section IV represents our findings in the form of a Multi-Valued Karnaugh Map (MVKM), while Section V compares our results to earlier work. Section VI concludes the paper.

II. Verbal and Mathematical Problem Description

Our problem is to analyze a commodity-supply system that was introduced in [1] and is shown in Fig. 1. The system has four pipelines that transmit a certain commodity (such as oil) from a single source point to three sink points (called stations). The status of pipeline i ($1 \leq i \leq 4$) is determined by a four-valued input variable X_i, which is described in Table 1(a). Likewise, the system status is determined by a four-valued output variables S described in Table 1(b). The indicator for meeting the demand of station i (called the success of station i) is given as a k_i-out-of-4: G system as shown in Table 2. The notation Sy(**A, X**) used in Table 2 denotes a symmetric switching function (SSF) of a characteristic set **A** and n arguments **X** where the number n of arguments is an implicit input of the function [7, 15, 16]. The four instances of the system variable S are related to station successes by [3]:

$$S\{0\} = \bar{S}_1 \quad (1a)$$

$$S\{1\} = S_1\bar{S}_2 \quad (1b)$$

$$S\{2\} = S_1 S_2\bar{S}_3 \quad (1c)$$

$$S\{3\} = S_1 S_2 S_3 \quad (1d)$$

III. System Analysis

We start the system analysis by encoding each of the multi-valued variables X_i ($1 \leq i \leq 4$) by two binary variables Z_{i1} and Z_{i2} as follows

$$X_i\{0\} = \bar{Z}_{i1}\bar{Z}_{i2} \quad (2a)$$

$$X_i\{1\} = \bar{Z}_{i1}Z_{i2} \quad (2b)$$

$$X_i\{2\} = Z_{i1}\bar{Z}_{i2} \quad (2c)$$

$$X_i\{3\} = Z_{i1} Z_{i2} \quad (2d)$$

Hence, the corresponding inputs needed in Table 2 are encoded as

$$X_i \{1\} \vee X_i \{2\} \vee X_i \{3\} = Z_{i1} \vee Z_{i2} \quad (3a)$$

$$X_i \{2\} \vee X_i \{3\} = Z_{i1} \quad (3b)$$

$$X_i \{3\} = Z_{i1} Z_{i2} \quad (3c)$$

Now, the successes of the three stations are given by

$$S_1 = Sy(\{4\}; Z_{11}\vee Z_{12}, Z_{21}\vee Z_{22}, Z_{31}\vee Z_{32}, Z_{41}\vee Z_{42})$$

$$= (Z_{11}\vee Z_{12})(Z_{21}\vee Z_{22})(Z_{31}\vee Z_{32})(Z_{41}\vee Z_{42}) \quad (4a)$$

$$S_2 = Sy(\{2, 3, 4\}; Z_{11}, Z_{21}, Z_{31}, Z_{41})$$

$$= Z_{11}Z_{21}\vee Z_{11}Z_{31}\vee Z_{11}Z_{41}\vee Z_{21}Z_{31}\vee Z_{21}Z_{41}\vee Z_{31} \quad (4b)$$

$$S_3 = Sy(\{3, 4\}; Z_{11}Z_{12}, Z_{21}Z_{22}, Z_{31}Z_{32}, Z_{41} Z_{42})$$

$$= Z_{11}Z_{12} Z_{21} Z_{22} Z_{31}Z_{32}\vee Z_{11}Z_{12} Z_{21} Z_{22} Z_{41}Z_{42}$$

$$\vee Z_{11}Z_{12} Z_{31} Z_{32} Z_{41}Z_{42} \vee Z_{21}Z_{22} Z_{31} Z_{32} Z_{41}Z_{42} \quad (4c)$$

The instances of S can now be obtained by combining equations (1) and (4). In the following, we state the values of these instances and their expectations

$$S\{0\} = \bar{S}_1 = \bar{Z}_{11}\bar{Z}_{12} \vee \bar{Z}_{21}\bar{Z}_{22} \vee \bar{Z}_{31}\bar{Z}_{32} \vee \bar{Z}_{41}\bar{Z}_{42}$$

$$= X_1\{0\}\vee X_2\{0\}\vee X_3\{0\}\vee X_4\{0\} \quad (5)$$

$$E\{S\{0\}\} = 1 - E\{\bar{X}_1\{0\}\}\, E\{\bar{X}_2\{0\}\}\, E\{\bar{X}_3\{0\}\}\, E\{\bar{X}_4\{0\}\} \quad (6)$$

Using properties of symmetric switching functions (SSFs) [7, 15, 16], we complement the expression (4b) for S_2 to obtain

$$\bar{S}_2 = Sy(\{0, 1\};\ Z_{11}, Z_{21}, Z_{31}, Z_{41})$$

$$= Sy(\{3, 4\};\ \bar{Z}_{11}, \bar{Z}_{21}, \bar{Z}_{31}, \bar{Z}_{41})$$

$$= \bar{Z}_{11}\bar{Z}_{21}\bar{Z}_{31} \vee \bar{Z}_{11}\bar{Z}_{21}\bar{Z}_{41}$$

$$\vee\ \bar{Z}_{11}\bar{Z}_{31}\bar{Z}_{41} \vee \bar{Z}_{21}\bar{Z}_{31}\bar{Z}_{41} \qquad (7)$$

Combining (4a) and (7), we express the next instance of the four-valued output as

$$S\{1\} = S_1\bar{S}_2 = (Z_{11} \vee Z_{12})(Z_{21} \vee Z_{22})(Z_{31} \vee Z_{32})(Z_{41} \vee Z_{42})$$

$$(\bar{Z}_{11}\bar{Z}_{21}\bar{Z}_{31} \vee \bar{Z}_{11}\bar{Z}_{21}\bar{Z}_{41} \vee \bar{Z}_{11}\bar{Z}_{31}\bar{Z}_{41} \vee \bar{Z}_{21}\bar{Z}_{31}\bar{Z}_{41})$$

$$= \bar{Z}_{11} Z_{12} \bar{Z}_{21} Z_{22} \bar{Z}_{31} Z_{32} (Z_{41}Z_{42} \vee Z_{41}\bar{Z}_{42} \vee \bar{Z}_{41}Z_{42})$$

$$\vee\ \bar{Z}_{11} Z_{12} \bar{Z}_{21} Z_{22} \bar{Z}_{41} Z_{42} (Z_{31}Z_{32} \vee Z_{31}\bar{Z}_{32} \vee \bar{Z}_{31}Z_{32})$$

$$\vee\ \bar{Z}_{11} Z_{12} \bar{Z}_{31} Z_{32} \bar{Z}_{41} Z_{42} (Z_{21}Z_{22} \vee Z_{21}\bar{Z}_{22} \vee \bar{Z}_{21}Z_{22})$$

$$\vee\ \bar{Z}_{21} Z_{22} \bar{Z}_{31} Z_{32} \bar{Z}_{41} Z_{42} (Z_{11}Z_{12} \vee Z_{11}\bar{Z}_{12} \vee \bar{Z}_{11}Z_{12})$$

$$= \bar{Z}_{11} Z_{12} \bar{Z}_{21} Z_{22} \bar{Z}_{31} Z_{32} (Z_{41}Z_{42} \vee Z_{41}\bar{Z}_{42})$$

$$\vee\ \bar{Z}_{11} Z_{12} \bar{Z}_{21} Z_{22} \bar{Z}_{41} Z_{42} (Z_{31}Z_{32} \vee Z_{31}\bar{Z}_{32})$$

$$\vee\ \bar{Z}_{11} Z_{12} \bar{Z}_{31} Z_{32} \bar{Z}_{41} Z_{42} (Z_{21}Z_{22} \vee Z_{21}\bar{Z}_{22})$$

$$\vee\ \bar{Z}_{21} Z_{22} \bar{Z}_{31} Z_{32} \bar{Z}_{41} Z_{42} (Z_{11}Z_{12} \vee Z_{11}\bar{Z}_{12})$$

$$\vee\ \bar{Z}_{11} Z_{12} \bar{Z}_{21} Z_{22} \bar{Z}_{31} Z_{32} \bar{Z}_{41} Z_{42} \qquad (8)$$

In equation (8), we note that for $(1 \le i \le 4)$ a certain literal of the variable Z_{i1} (either $\bar{Z}_{i1}$ or Z_{i1}) is always paired (ANDed) with a literal of the variable Z_{i2} (either $\bar{Z}_{i2}$ or Z_{i2}). Therefore, we can decode such pairs of literals into the corresponding instance of multi-valued variable X_i using (2), namely

$$S\{1\} = X_1\{1\}\ X_2\{1\}\ X_3\{1\}\ (X_4\{3\} \vee X_4\{2\})$$

$$\vee\ X_1\{1\}\ X_2\{1\}\ X_4\{1\}\ (X_3\{3\} \vee X_3\{2\})$$

$$\vee\ X_1\{1\}\ X_3\{1\}\ X_4\{1\}\ (X_2\{3\} \vee X_2\{2\})$$

$$\vee\ X_2\{1\}\ X_3\{1\}\ X_4\{1\}\ (X_1\{3\} \vee X_1\{2\})$$

$$\vee\ X_1\{1\}\ X_2\{1\}\ X_3\{1\}\ X_4\{1\} \qquad (9)$$

Equation (9) is a probability-ready expression [17-23] in which ANDed entities are independent and ORed entities are disjoint, and hence any of its entities are replaced by its expectation, while logical multiplication (ANDing) and logical addition (ORing) are replaced by their arithmetic counterparts, namely

$$E\{S\{1\}\} = E\{X_1\{1\}\}\ E\{X_2\{1\}\}\ E\{X_3\{1\}\}\ (E\{X_4\{3\}\} + E\{X_4\{2\}\})$$

$$+\ E\{X_1\{1\}\}\ E\{X_2\{1\}\}\ E\{X_4\{1\}\}\ (E\{X_3\{3\}\} + E\{X_3\{2\}\})$$

$$+\ E\{X_1\{1\}\}\ E\{X_3\{1\}\}\ E\{X_4\{1\}\}\ (E\{X_2\{3\}\} + E\{X_2\{2\}\})$$

$$+\ E\{X_2\{1\}\}\,E\{X_3\{1\}\}\ \ E\{X_4\,\{1\}\}\,(E\{X_1\{3\}\}+E\{X_1\{2\}\})$$

$$+\ E\{X_1\{1\}\}\,E\{X_2\,\{1\}\}\ \ E\{X_3\,\{1\}\}\,E\{X_4\{1\}\} \qquad (10)$$

Next, we prove that

$$S_1S_3 \leq\ S_1S_2\,, \qquad (11)$$

which is a useful result, since it simplifies our expression for S{3} to

$$S\{3\} = S_1\,S_2\,S_3 =\ S_1\,S_3 \qquad (12)$$

First, we compute $S_1\,S_3$ as

$$S_1\,S_3 =\ (\,Z_{11}\vee Z_{12}\,)\,(\,Z_{21}\vee Z_{22}\,)\,(\,Z_{31}\vee Z_{32}\,)\,(\,Z_{41}\vee Z_{42}\,)$$

$$(\,Z_{11}Z_{12}\,Z_{21}Z_{22}Z_{31}Z_{32}$$

$$\vee\ Z_{11}Z_{12}\,Z_{21}Z_{22}Z_{41}Z_{42}$$

$$\vee\ Z_{11}Z_{12}\,Z_{31}Z_{32}Z_{41}Z_{42}$$

$$\vee Z_{21}Z_{22}\,Z_{31}Z_{32}Z_{41}Z_{42})$$

$$= Z_{11}Z_{12}\,Z_{21}Z_{22}\,Z_{31}Z_{32}\ \ (Z_{41}\vee Z_{42}\,)$$

$$\vee\ Z_{11}Z_{12}\,Z_{21}Z_{22}Z_{41}Z_{42}\,(Z_{31}\vee Z_{32})$$

$$\vee\ Z_{11}Z_{12}Z_{31}Z_{32}Z_{41}Z_{42}\ \ (Z_{21}\vee Z_{22}\,)$$

$$\vee\ Z_{21}\ \,Z_{22}\,Z_{31}\,Z_{32}Z_{41}\ \,Z_{42}\ \ (Z_{11}\vee Z_{12}\,)$$

$$=\ Z_{11}Z_{12}\ \ Z_{21}Z_{22}\,Z_{31}Z_{32}\ \ Z_{41}\vee Z_{42}$$

$$\vee\ Z_{11}Z_{12}\,Z_{21}\ \,Z_{22}Z_{31}\ \,Z_{32}\ \ (Z_{41}\bar{Z}_{42}\vee\ \bar{Z}_{41}Z_{42}\,)$$

$$\vee\ Z_{11}Z_{12}\,Z_{21}Z_{22}Z_{41}Z_{42}\ \ (Z_{31}\bar{Z}_{32}\vee\ \bar{Z}_{31}Z_{32}\,)$$

$$\vee\ Z_{11}Z_{12}\ \ Z_{31}\,Z_{32}\,Z_{41}Z_{42}\ \ \ (Z_{21}\bar{Z}_{22}\vee\ \bar{Z}_{21}Z_{22}\,)$$

$$\vee\ Z_{21}Z_{22}\,Z_{31}\ \,Z_{32}Z_{41}Z_{42}\,(Z_{11}\bar{Z}_{12}\vee\ \bar{Z}_{11}Z_{12}\,) \qquad (13)$$

Equation (13) shows that S_1S_3 is a disjunction of nine minterms over the eight variables $(\,Z_{i1}\,,\ Z_{i2}\,)\,,\ 1\ \leq i\ \leq 4\,.$

Now, we compute S_1S_2as

$$S_1\,S_2 = (\,Z_{11}\vee Z_{12}\,)\,(\,Z_{21}\vee Z_{22}\,)\,(\,Z_{31}\vee Z_{32}\,)\,(\,Z_{41}\vee Z_{42}\,)$$

$$(\,Z_{11}Z_{21}\vee\ Z_{11}Z_{31}\vee\ Z_{11}Z_{41}\vee\ Z_{21}Z_{31}\vee\ Z_{21}Z_{41}\vee\ Z_{31}Z_{41}\,) \qquad (14)$$

It is clear that each of the minterms in (13) subsumes some term in S_1S_2 (when the latter is expanded in SOP form). Hence, Equation (11) follows. We note that pairing of literals observed in (8) also occurs in (13), and hence we can enforce a similar decoding using (2) to obtain

$$S\{3\} = S_1\,S_3 = X_1\{3\}\ X_2\,\{3\}\ X_3\,\{3\}X_4\{3\}$$

$$\vee\ X_1\{3\}\ X_2\ \{3\}\ X_3\{3\}\ (X_4\{2\} \vee X_4\{1\})$$

$$\vee\ X_1\{3\}\ X_2\ \{3\}\ X_4\{3\}\ (X_3\{2\} \vee\ X_3\{1\})$$

$$\vee\ X_1\{3\}\ X_3\{3\}\ X_4\{3\}\ (X_2\{2\} \vee X_2\{1\})\ \ \vee\ \ X_2\{3\}\ X_3\{3\}\ X_4\{3\}\ (X_1\{2\} \vee X_1\{1\}) \qquad (15)$$

Now, Equation (15) is in PRE form, and can be rewritten as

$$E\{S\{3\}\} = E\{X_1\{3\}\}\ E\{X_2\{3\}\}E\{X_3\{3\}\}E\{X_4\{3\}\}$$

$$+E\{X_1\{3\}\}\ E\{X_2\{3\}\}\ E\{X_3\ 3\}\}\ (E\{X_4\{2\}\}\ +E\{X_4\{1\}\})$$

$$+\ E\{X_1\{3\}\}\ \ E\{X_2\{3\}\}\ E\{X_4\{3\}\}\ (E\{X_3\{2\}\}+E\{X_3\{1\}\})$$

$$+\ E\{X_1\{3\}\}\ \ E\{X_3\{3\}\}\ E\{X_4\{3\}\}\ (E\{X_2\{2\}\}\ +E\{X_2\{1\}\})$$

$$+\ E\{X_2\{3\}\}\ E\{X_3\{3\}\}\ E\{X_4\{3\}\}\ (E\{X_1\{2\}\}+E\{X_1\{1\}\}) \qquad (16)$$

It is now possible to compute S{2} *via* (1c), but we opt for a shortcut by observing that the four instances S{0}, S{1}, S{2}, and S{3} form an orthonormal set, that is, for every input configuration one and only one of these four values is 1 while each of the remaining three is 0. In particular, we obtain the expectation of S{2} as

$$E\{S\{2\}\} = 1 - (E\{S\{0\}\} + \ E\{S\{1\}\} + E\{S\{3\}\}) \qquad (17)$$

IV. MVKM Construction

Based on the analysis of Section III, we construct the Multi-Valued Karnaugh Map (MVKM) of Fig. 2. This map is the natural map representing our multi-state system, and various variants of it have been published before [24-26]. The map variables are the four input four-valued variables $X_i\{1 \le i \le 4\}$. Since each of these variables can take four values independently of the other variables, the number of map cells is $4^4 = 256$. A map entry is one of the four values {0, 1, 2, 3} that the output variable takes. The entry of a map cell is chosen as one of the values 0, 1, and 3 according to equations (5), (9), and (15), respectively. The remaining cells are entered with the value 2.

The MVKM of Fig.2 might be viewed as a map of the eight binary variables (Z_{i1}, Z_{i2}), $\{1 \le i \le 4\}$. For convenience, these variables are superimposed as alternative map variables in Fig. 2. According to this point of view, the number of map cells should be 2^8, *i.e.,* it remains equal to 256. The number of map sells is large, indeed, This constitutes a particular source of difficulty, which is partially remedied through the use of a conveniently regular and scalable map structure [27-29]. The map is very convenient for decomposing its function into sub-functions, *i.e.*, for embedding an expansion tree or a decision diagram for this function [15, 16, 23, 30].

Coherence of the present multi-state system is illustrated neatly by the MVKM. In fact, the three properties of coherence can be observed from the MVKM (reproduced in Fig.3) as follows:

1. **Causality** is evident from the map since the entry of the shaded all-0 cell is 0 (indicating that S(0, 0, 0, 0) = 0) and the entry of the dotted all-3 cell is 3 (indicating that S (3, 3, 3, 3) = 3).

2. **Monotonicity** with represent to component 1 can be observed by dividing the map into four quarters each consisting of four map columns. These four quarters represent the regions of $X_1 = 0$, $X_1 = 1$, $X_1 = 3$, and $X_1 = 2$. The map in Fig. 3 is divided vertically by a primary mirror into two halves: a lower half comprised of the regions $X_1 = 0$ and $X_1 = 1$, and a higher half comprised of the regions $X_1 = 2$ and $X_1 = 3$. The entry of any cell in the higher region is equal to or greater than the entry of its lower-region image with respect to the primary mirror. This means that when the values of X_2, X_3 and X_4 are fixed

$$S \mid X_1 = 3 \ge S \mid X_1 = 1 \qquad (18a)$$

$$S \mid X_1 = 2 \ \ge S \mid X_1 = 0 \qquad (18b)$$

Each of the two halves of the map is divided into two quarters by a secondary mirror. The entry of a cell in the quarter of $\{X_1 = 0\}$ is equal to or greater than the entry of its image (w.r.t. the left secondary image mirror) in the $\{X_1 = 0\}$ quarter. This means that for fixed values of X_2, X_3, and X_4, one has

$$S \mid X_1 = 1 \geq S \mid X_1 = 0 \tag{18c}$$

likewise, the entry of a cell in the quarter $\{X_1 = 3\}$ is equal to or greater than the entry of its image (w.r.t. the right secondary image mirror) in the quarter$\{X_1 = 2\}$. This means that for fixed values of X_2, X_3, and X_4 we have

$$S \mid X_1 = 3 \geq S \mid X_1 = 2 \tag{18d}$$

Combining equations (18a) – (18d), we obtain for fixed values of X_2, X_3, and X_4, the following assertion that the output S is non-decreasing when X_1 increases

$$S \mid X_1 = 3 \geq S \mid X_1 = 2 \geq S \mid X_1 = 1 \geq S \mid X_1 = 0 \tag{19}$$

The MVKM can be similarly used to demonstrate monotonicity w.r.t. each of the remaining components 2, 3, and 4.

3. **Relevancy** of component 1 is evident from the fact that there are cases in which the inequalities in (19) are strict ones. For example, the map row labelled 31 (for $X_3 = 3$ and $X_4 = 1$) and coloured in yellow has two starred cells for $X_2 = 3$ in which

$$S \mid X_1 = 3 > S \mid X_1 = 2 \tag{20a}$$

with two double-starred cells for $X_2 = 1$ in which

$$S \mid X_1 =2 > S \mid X_1 =1 \tag{20b}$$

and three pairs of cells with the same X_2 values for which

$$S \mid X_1 = 1 > S \mid X_1 = 0 \tag{20c}$$

The map can be used in a similar fashion to demonstrate relevancy of each of the components 2, 3, and 4.

V. Comparison with Previous Work

The problem handled herein was solved by multi-state techniques by Tian *et al.* [1]. And Mo. *et al.* [2], and was also solved *via* techniques of multi-valued logic by Rushdi [3], and *via* switching-algebraic techniques by Rushdi and Al-Amoudi [4]. In all cases the following input was used

$$\{E\{X_i\{j\}\}\}=\begin{bmatrix} .050 & .0950 & .0684 & .7866 \\ .050 & .0950 & .0684 & .7866 \\ .030 & .0776 & .0446 & .8478 \\ .030 & .0776 & .0446 & .8478 \end{bmatrix} \quad (1\leq i \leq 4,\ 0 \leq j \leq 3) \tag{21}$$

Table 3 compares our results for this specific input with the results of earlier three teams of authors. The four sets of results are essentially the same, despite the existence of differences in precision. We deliberately use an exaggerated precision of fifteen significant digits for comparison purposes.

VI. Conclusions

This paper demonstrated how the reliability of a multi-state system can be analyzed *via* multi-valued logic. Detailed solution for a standard commodity-supply multi-state system was provided, and culminated in a MVKM for the multi-state system. The solution obtained was satisfactorily checked against previously reported work.

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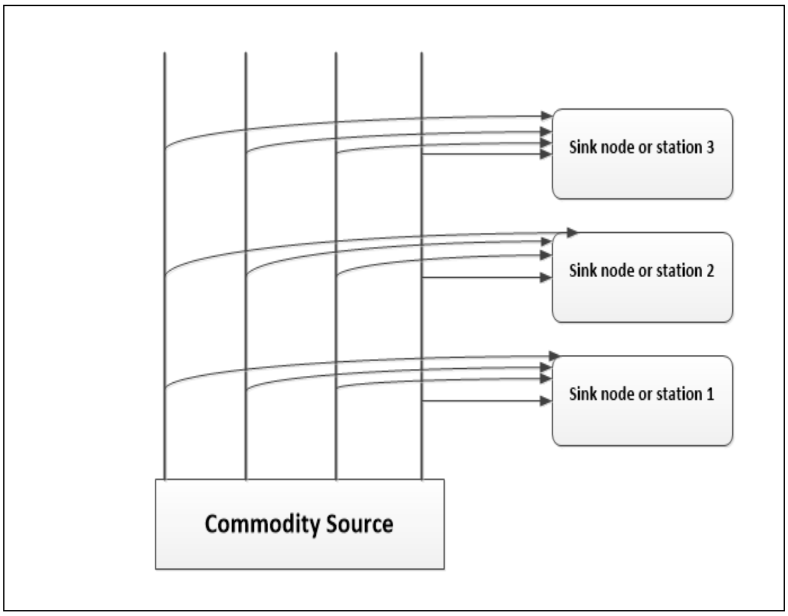

Fig. 1. A commodity-supply system that is modeled as a multi-state k-out-of-n: G system (Adapted from *Tian et al.* [1]).

X_3X_4 \ X_1X_2	00	01	03	02	12	13	11	10	30	31	33	32	22	23	21	20
00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
01	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	2	2	1	0	0	2	2	2	2	2	2	0
13	0	0	0	0	2	2	1	0	0	2	3	2	2	2	2	0
11	0	0	0	0	1	1	1	0	0	1	2	2	2	2	1	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	2	2	1	0	0	2	3	2	2	2	2	0
33	0	0	0	0	2	3	2	0	0	3	3	3	2	3	2	0
32	0	0	0	0	2	2	2	0	0	2	3	2	2	2	2	0
22	0	0	0	0	2	2	2	0	0	2	2	2	2	2	2	0
23	0	0	0	0	2	2	2	0	0	2	3	2	2	2	2	0
21	0	0	0	0	2	2	1	0	0	2	2	2	2	2	2	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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Fig. 2. A Multi-Valued Karnaugh Map (MVKM) that might be viewed as

(a) a map of eight binary variables Z_{i1}, $Z_{i2}\{1 \leq i \leq 4\}$, or

(b) a map of four four-valued variables $X_i\{1 \leq i \leq 4\}$

X_3X_4 \ X_1X_2	$X_1=0$				$X_1=1$				$X_1=3$				$X_1=2$			
	00	01	03	02	12	13	11	10	30	31	33	32	22	23	21	20
00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
01	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	2	2	1	0	0	2	2	2	2	2	2	0
13	0	0	0	0	2	2	1	0	0	2	3	2	2	2	2	0
11	0	0	0	0	1	1	1	0	0	1	2	2	2	2	1	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	2	2	1**	0	0	2	3*	2	2	2*	2**	0
33	0	0	0	0	2	3	2	0	0	3	3	3	2	3	2	0
32	0	0	0	0	2	2	2	0	0	2	3	2	2	2	2	0
22	0	0	0	0	2	2	2	0	0	2	2	2	2	2	2	0
23	0	0	0	0	2	2	2	0	0	2	3	2	2	2	2	0
21	0	0	0	0	2	2	1	0	0	2	2	2	2	2	2	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Secondary Mirror Primary Mirror Secondary Mirror

S

Fig. 3. The MVKM in Fig. 2 reproduced to demonstrate coherency properties.

Table 1a. Defintion of the four-valued input variable X_i, which deteminesthe status of pipeline i ($1 \leq i \leq 4$)

Value of X_i	Meaning
0	Pipeline i does not transmit the commodity to any station.
1	Pipeline i transmits the commodity to station 1.
2	Pipeline i transmits the commodity to stations 1 and 2.
3	Pipeline i transmits the commodity to stations 1, 2, and 3.

Table 1b. Description of of the four-valued output variable S, which detemines system status.

Value of S	Meaning
0	The system does not meet the commodity demand of any station.
1	The system meets the commodity demand of station 1.
2	The system meets the commodity demand of stations 1 and 2.
3	The system meets the commodity demand of any stations 1, 2, and 3.

Table 2. Description of the three stations, each as a k_m-out-of-4: G system

Station m	The value of k_m as a k-out-of-4: G system	The four inputs	Station Success
1	4	$(X_i\{1\} \vee X_i\{2\} \vee X_i\{3\})$ $1 \le i \le 4$	$S_1 = Sy(\{4\}; X_1\{1\} \vee X_1\{2\} \vee X_1\{3\}, X_2\{1\} \vee X_2\{2\} \vee X_2\{3\}, X_3\{1\} \vee X_3\{2\} \vee X_3\{3\}, X_4\{1\} \vee X_4\{2\} \vee X_4\{3\})$
2	2	$(X_i\{2\} \vee X\{3\})$ $1 \le i \le 4_i\{3\}$	$S_1 = Sy(\{2, 3, 4\}; X_1\{2\} \vee X_1\{3\}, X_2\{2\} \vee X_2\{3\}, X_3\{2\} \vee X_3\{3\}, X_4\{2\} \vee X_4\{3\})$
3	3	$X_i\{3\}$ $1 \le i \le 4_i\{3\}$	$S_1 = Sy(\{3, 4\}; X_1\{3\}, X_2\{3\}, X_3\{3\}, X_4\{3\})$

Table 3. Comparison of our results with earlier work

Expectation of	Tian *et al.*[1]	Mo *et al.*[2]	Rushdi [3]	Rushdi & Al-Amoudi [4]	Our results
S(0)	**0.1508**	**0.150838**	**0.150837750000**	**0.150837750000000**	**0.150837750000000**
S(1)	**0.0023**	**0.002282**	**0.002282548128**	**0.002282548128000**	**0.002282548128000**
S(2)	**0.0892**	**0.089181**	**0.089180866436**	**0.089180866435691**	**0.089180866435691**
S(3)	**0.7577**	**0.757699**	**0.757698835436**	**0.757698835436309**	**0.757698835436309**
Total	**1.0000**	**1.000000**	**1.000000000000**	**1.000000000000000**	**1.000000000000000**